Int J Theor Phys (2009) 48: 2353–2364
DOI 10.1007/s10773-009-0024-z

Geometric Phase in Fluctuating Magnetic Field

Z.S. Wang

Received: 19 February 2009 / Accepted: 19 April 2009 / Published online: 28 April 2009

Abstract Geometric phase in a two-level atom with a fluctuating magnetic field is calculated by a nonunit vector ray in a complex projective Hilbert space, where the nonunit vector is a map connecting with density matrices of a quantum open system. We find that the Pancharatnam phase oscillates with evolving time. The Berry phase depends on the fluctuating parameter but it is proportional to the area spanned in the Bloch parameter space.

Keywords Geometric phase · Open system · Fluctuation magnetic field

1 Introduction

The wave function of a quantum system retains a memory of its motion in terms of a geometric phase [1–7] when it undergoes a closed evolution in parameter space: the geometric phase essentially arises as an effect of parallel transport in the Poincaré representation of the manifold of pure states. More precisely, the amplitudes of wave functions are mapped onto given points on the Poincaré sphere for a pure state [8–10]. Geometric phase has been observed in spin $1/2$ systems through nuclear-magnetic-resonance (NMR) experiments [11] and with polarized photons using interferometers (PPI) [12, 13].

As early as 1956, Pancharatnam [1] anticipated a quantum geometric phase. Surprisingly nobody paid much attention to the existence of the geometric phase until 1984 when Berry [2] rediscovered the phase for a quantum system under adiabatic evolution. Aharonov and Anandan [3] subsequently regarded the geometric phase factor as an inherent geometric property of the quantum system itself in the evolution and dropped the requirement for adiabatic condition.

Application of geometric phases in quantum computation [14–16] has motivated their studies under more realistic situations [17–30]. In a real system, noise and decoherence

Z.S. Wang (✉)
College of Physics and Communication Electronics, Jiangxi Normal university, Nanchang 33022, P.R. China
e-mail: zishengwang@yahoo.com

are big problems. The process limits the ability to maintain pure quantum states in quantum information. It is known that the decoherence may come from the fluctuations of both physical quantities and vacuums. In previous works [18–29], however, almost studies have concentrated to the effects of vacuum fluctuations on the geometric phase. Therefore, it is interesting to study the effects of fluctuating magnetic field on the geometric phase. It may be very important, especially, for process of quantum information because noisy mixing of matter would also mimic a fluctuating matter. Differently from the previous classic treatment about the fluctuating magnetic field [30], we will start from a master equation in order to investigate the effects of fluctuating magnetic field on the geometric phase.

It is known that the definition of the geometric phase for the open system is still an open problem. We formulate entirely the geometric phase in terms of geometric structures on a complex projective Hilbert space because a general belief is that the Berry phases are geometric in their nature, i.e., proportional to the area spanned in parameter space.

2 Master Equation in Fluctuating Magnetic Field

Our system consists of a two-level atom in the presence of an external magnetic field with the Hamiltonian $H_0 = \frac{1}{2}\hbar\Omega\sigma_z$, where Ω is the atomic resonance frequency. We further extend our analysis to contain both the magnetic field and a fluctuating component. Thus a master equation in the Schrödinger picture may be expressed by a density operator ρ,

$$i\hbar\frac{d}{dt}\rho(t) = [H(t), \rho(t)], \tag{1}$$

where the Hamiltonian is taken to be linear in a fluctuating field and expressed as

$$H(t) = H_0 + B(t)M, \tag{2}$$

where $B(t)$ is a random field to describe a decoherence source in the physical system and $M = \frac{1}{2}\hbar\sigma_x$ is independent of time. After averaging on different trajectories induced by the noise, actually, the system at the end of the evolution is in a mixed state. Thus the Bloch vector does not return to its initial position since the state has changed its degree of mixedness. To calculate the correct Bloch vector, we have to average the final positions. In the interaction picture, the master equation (1) may be written as

$$i\hbar\frac{d}{dt}\rho_I(t) = [H_I(t), \rho_I(t)], \tag{3}$$

where $\rho_I(t) = (\mathcal{P}e^{iH_0t/\hbar})\rho(t)(\mathcal{P}e^{-iH_0t/\hbar})$ and $H_I(t) = B(t)(\mathcal{P}e^{iH_0t/\hbar})M(\mathcal{P}e^{-iH_0t/\hbar}) = B(t)M_I$, while $\mathcal{P}$ is a time-ordering operator. This equation can be solved by iterating as [31]

$$\begin{aligned}\rho_I(t) &= \rho_I(0) - \frac{i}{\hbar}\int_0^t dt_1 B(t_1)[M_I(t_1), \rho_I(0)] \\ &\quad - \frac{1}{\hbar^2}\int_0^t\int_0^{t_1} dt_1 dt_2 B(t_1)B(t_2)[M_I(t_1), [M_I(t_2), \rho_I(0)]] + \cdots.\end{aligned} \tag{4}$$

In order to simplify our computation but without loss of generality, we now assume that the fluctuations in such fields may be well approximated by random added to an average value,

$$\langle B(t)\rangle = 0, \qquad \langle B(t_1)B(t_2)\rangle = \alpha^2\tau\delta(t_1 - t_2), \tag{5}$$

with the average of all higher odd products of $B(t)$ vanishing, and all higher even products given by the sum of all possible products of second order cumulants. For example, $\langle B(t_1)B(t_2)B(t_3)B(t_4)\rangle = \frac{1}{4}\alpha^4\tau^2(\delta(t_1-t_2)\delta(t_3-t_4)+\delta(t_1-t_3)\delta(t_2-t_4)+\delta(t_1-t_4)\times\delta(t_3-t_2))$. It is noted that the random potentials are supposed to δ-correlated Gaussian distributions, which leads to only the even-product contributions to the average density $\langle\rho(t)\rangle$.

In (4), thus, the first term produces the unperturbed pure state, the second (dipole) term vanishes after averaging over the fluctuations. Moreover, the quadrupole term gives the leading fluctuation-induced modification, and so on.

Averaged values of functions of the random field $B(t)$ can then be expressed as a path integral,

$$\langle g(B(t))\rangle = \int_{-\infty}^{\infty} D[B(t)]g(B)e^{-\int_0^t dt B^2/2k}, \tag{6}$$

where

$$D[B(t)] = \Pi_i dB(t_i)\sqrt{\frac{\Delta t}{2k\pi}}, \tag{7}$$

and $k = 2\alpha^2\tau$. This reproduces the form of the averages of the even power products, and integrals can be explicitly calculated with the result,

$$\begin{aligned}\langle\rho_I(t)\rangle &= \langle\rho_I(0)\rangle - \frac{k(t)}{2\hbar^2}\int_0^t dt_1[M_I(t_1),[M_I(t_1),\langle\rho_I(0)\rangle]]\\ &\quad + \frac{k^2(t)}{4\hbar^4}\int_0^t dt_1\left[M_I(t_1),\left[M_I(t_1),\int_0^{t_1}[M_I(t_2),[M_I(t_2),\langle\rho_I(0)\rangle]]\right]\right] - \cdots,\end{aligned} \tag{8}$$

which is just the iterative expression of the following differential equation,

$$\frac{d}{dt}\langle\rho_I(t)\rangle = -\frac{k(t)}{2\hbar^2}[M_I(t),[M_I(t),\langle\rho_I(t)\rangle]]. \tag{9}$$

In the Schrödinger picture, thus, (9) becomes

$$\frac{d}{dt}\langle\rho(t)\rangle = -\frac{i}{\hbar}[H_0,\langle\rho(t)\rangle] - \frac{k(t)}{2\hbar^2}[M(t),[M(t),\langle\rho(t)\rangle]], \tag{10}$$

where $\langle\cdots\rangle$ represents an average over the random variables. The double commutator is the *CPT* violating term since although it is *CP* symmetric it induces time-irreversibility, which may be an appropriate form for the decoherence. It is noted that (10) is different from the others from a system interacting with the normal [27, 32] and squeezed vacuum fluctuation [29].

Defining $u(t) = (\langle\rho_{11}\rangle - \langle\rho_{22}\rangle)/2$, $v(t) = 2\mathrm{Re}(\langle\rho_{12}\rangle)$ and $w(t) = 2\mathrm{Im}(\langle\rho_{12}\rangle)$, one find that (10) may be rewritten as

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}u\\ v\\ w\end{pmatrix} &= \begin{pmatrix}-k/2 & 0 & 0\\ 0 & 0 & -\Omega\\ 0 & \Omega & -k/2\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}\\ &= \mathcal{L}\begin{pmatrix}u\\ v\\ w\end{pmatrix},\end{aligned} \tag{11}$$

which is similar to the optical Bloch equation and $\mathcal{L}$ is a superoperator.

3 Bloch Vector

In general case, the random fluctuation of magnetic field is small in comparison with the atom oscillation length. Therefore, we only consider the case of $\Omega_r^2 = \Omega^2 - \kappa^2 \geq 0$, where $\kappa = k/4$ will be taken as the constant for the sake of simplification. In order to obtain the solution of (11), we firstly calculate the eigenvectors $\overrightarrow{E}^{(i)}$ and corresponding eigenvalues λ_i of the superoperator $\mathcal{L}$. It is straightforward to show that the eigenvectors are given by,

$$\overrightarrow{E}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \overrightarrow{E}^{(2)} = \begin{pmatrix} 0 \\ -\frac{-\kappa+i\Omega_r}{\Omega} \\ 1 \end{pmatrix}, \qquad \overrightarrow{E}^{(3)} = \begin{pmatrix} 0 \\ -\frac{-\kappa-i\Omega_r}{\Omega} \\ 1 \end{pmatrix}, \tag{12}$$

and the corresponding eigenvalues are expressed by,

$$\lambda_1 = -2\kappa, \qquad \lambda_2 = -\kappa - i\Omega_r, \qquad \lambda_3 = -\kappa - i\Omega_r. \tag{13}$$

Thus, the solution of (11) is given by

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = c_1 e^{\lambda_1 t}\overrightarrow{E}^{(1)} + c_2 e^{\lambda_2 t}\overrightarrow{E}^{(2)} + c_3 e^{\lambda_3 t}\overrightarrow{E}^{(3)}, \tag{14}$$

where c_i $(i = 1, 2, 3)$ are determined by taking the initial state of the two-level atom as $|\Psi(0)\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$. Thus the diagonal elements of the density matrix can be expressed by

$$\langle\rho_{11}\rangle = \frac{1}{2}(1 + \cos\theta e^{-2\kappa t}), \tag{15}$$

$$\langle\rho_{22}\rangle = \frac{1}{2}(1 - \cos\theta e^{-2\kappa t}). \tag{16}$$

From (15) and (16), we see that the diagonal elements include an effect of decay, which is similar to the spontaneous decay of an atom interacting with the normal vacuum reservoir.

For the nondiagonal elements of the density matrix, one finds

$$\begin{aligned} \langle\rho_{12}\rangle &= \frac{1}{4}\sin\theta\left[-i\frac{\kappa}{\Omega_r} + \left(1 + \frac{\Omega}{\Omega_r}\right)\right]e^{-\kappa t - i\Omega_r t} \\ &\quad + \frac{1}{4}\sin\theta\left[i\frac{\kappa}{\Omega_r} + \left(1 - \frac{\Omega}{\Omega_r}\right)\right]e^{-\kappa t + i\Omega_r t}, \end{aligned} \tag{17}$$

$$\begin{aligned} \langle\rho_{21}\rangle &= \frac{1}{4}\sin\theta\left[-i\frac{\kappa}{\Omega_r} + \left(1 - \frac{\Omega}{\Omega_r}\right)\right]e^{-\kappa t - i\Omega_r t} \\ &\quad + \frac{1}{4}\sin\theta\left[i\frac{\kappa}{\Omega_r} + \left(1 + \frac{\Omega}{\Omega_r}\right)\right]e^{-\kappa t + i\Omega_r t}, \end{aligned} \tag{18}$$

which include both the complex oscillations and exponential decay with the evolving time. When the fluctuating parameter $\kappa = 0$, the oscillations become simple with the frequency Ω related to the magnetic field.

Using the elements of density matrix in (15)–(18), we can parameterize the Bloch vector $\mathbf{n} = \mathrm{Tr}(\langle\rho\rangle\overrightarrow{\sigma}) = (\langle\rho_{12}\rangle + \langle\rho_{21}\rangle, i(\langle\rho_{12}\rangle - \langle\rho_{21}\rangle), \langle\rho_{11}\rangle - \langle\rho_{22}\rangle)$ as $\mathbf{n} = (r\sin\alpha\cos\beta, r\sin\alpha\sin\beta, r\cos\alpha)$ with the relations,

$$\begin{aligned} r^2 = \mathbf{n}\cdot\mathbf{n} &= ((\langle\rho_{12}\rangle + \langle\rho_{21}\rangle)^2 - (\langle\rho_{12}\rangle - \langle\rho_{21}\rangle)^2 + (\langle\rho_{11}\rangle - \langle\rho_{22}\rangle)^2 \\ &= \sin^2\theta\left(\left(\cos\Omega_r t - \frac{\kappa}{\Omega_r}\sin\Omega_r t\right)^2 + \frac{\Omega^2}{\Omega_r^2}\sin^2\Omega_r t\right)e^{-2\kappa t} + \cos^2\theta e^{-4\kappa t}, \end{aligned} \tag{19}$$

where r is radius of the Bloch sphere. And two azimuthal angles, α and β, can be expressed by

$$\cos\alpha = \frac{\langle\rho_{11}\rangle - \langle\rho_{22}\rangle}{r} = \frac{\cos\theta}{r}e^{-2\kappa t}, \tag{20}$$

$$\tan\beta = i\frac{\langle\rho_{12}\rangle - \langle\rho_{21}\rangle}{\langle\rho_{12}\rangle + \langle\rho_{21}\rangle} = \frac{\Omega\sin\Omega_r t}{\Omega_r(\cos\Omega_r t - \frac{\kappa}{\Omega_r}\sin\Omega_r t)}. \tag{21}$$

From (19)–(21), we see that the Bloch parameters, such as r, α and β, depend on the fluctuating rate and oscillate with the evolving time. Especially, β doesn't depend on the exponential decay.

4 Geometric Phase

When $r = 1$, our physical system corresponds to a pure state. It is well-known that the given points on the unit Poincaré sphere mapped onto field amplitudes as the pure states $|\psi\rangle$ are unit vectors in the complex projective Hilbert space [8–10]. For $r < 1$, the physical system corresponds to a mixed state. The corresponding $\mathbf{n} < \mathbf{n}^R = \mathbf{n}/r$ are in interior of this unit Poincaré sphere. The mixed states, therefore, can be identified with the interior points of this sphere. Thus, the interior points in the unit Poincaré sphere may be mapped onto field amplitudes as the mixed states $|\Psi\rangle$, which can be expressed as

$$\Pi^{-1}(\rho) = |\Psi\rangle = \begin{pmatrix} \sqrt{r}\cos(\alpha/2) \\ \sqrt{r}e^{i\beta}\sin(\alpha/2) \end{pmatrix}, \tag{22}$$

which includes the effect of fluctuating magnetic field. Therefore, it is a dressed spin $-1/2$ state with nonunit vector in the complex projective Hilbert space.

Our nonunit vector ray in the projective Hilbert space is one-to-one correspondence with the general density matrix evolved dynamically according to (10) [27]. In the kinematic approach to the geometric phase [33], the state vector of the mixed state was taken as a purified state. It is well-known that the purification of the density operator is not usually unique.

We now begin with the smooth (open or closed) curve $\mathcal{C} = \{|\Psi(t)\rangle = \Pi^{-1}\rho(t)\}$ and subdivide it into N parts. The points of subdivision are at $t_0 = 0, t_1, \ldots, t_N = t$ and $|\Psi_i\rangle = |\Psi(t_i)\rangle = \Pi^{-1}\rho(t_i)$ are values at these points. Each trajectory, then, is represented by a discrete sequence of quantum states $\{|\Psi_0\rangle, |\Psi_1\rangle, \ldots, |\Psi_N\rangle\}$. Thus the geometric phase, expressed by the wave function in the complex projective Hilbert space, is given by the

Pantcharatnam formula,

$$
\begin{aligned}
\gamma_g &= -\mathcal{L}t_{N\to\infty} \arg\{\langle\Psi_0|\Psi_1\rangle\langle\Psi_1|\Psi_2\rangle\cdots\langle\Psi_{N-1}|\Psi_N\rangle\langle\Psi_N|\Psi_0\rangle\} \\
&= \arg\left(\langle\Psi(t_0)|\Psi(t)\rangle \exp\left[-\left(\int_{t_0}^{t} \frac{\langle\Psi(t)|d|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle}\right)\right]\right) \\
&= \arg\langle\Psi(t_0)|\Psi(t)\rangle - \mathrm{Im}\left(\int_{t_0}^{t} \frac{\langle\Psi(t)|d|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle}\right) \\
&= \arg\langle\Psi(t_0)|\Psi(t)\rangle - \frac{1}{2}\int_{t_0}^{t}(1-\cos\alpha(t))d\beta(t),
\end{aligned}
\tag{23}
$$

where the total phase is given by

$$
\arg\langle\Psi(t_0)|\Psi(t)\rangle = \tan^{-1}\frac{\sin(\beta(t)-\beta(t_0))\sin\frac{\alpha(t_0)}{2}\sin\frac{\alpha(t)}{2}}{\cos\frac{\alpha(t_0)}{2}\cos\frac{\alpha(t)}{2}+\cos(\beta(t)-\beta(t_0))\sin\frac{\alpha(t_0)}{2}\sin\frac{\alpha(t)}{2}}. \tag{24}
$$

Equation (23) is a gauge and reparameterized invariance. Therefore, γ_g is a geometric (Pantcharatnam) phase associated with an evolution of a quantum open system.

By setting

$$
|\Psi^p(t)\rangle = \exp\left\{-\left(\int_{t_0}^{t} \frac{\langle\Psi(t)|d|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle}\right)\right\}|\Psi(t)\rangle, \tag{25}
$$

which satisfies the parallel transport condition, we find that the Pantcharatnam phase (23) may be rewritten as

$$
\gamma_g = \arg\langle\Psi^p(t_0)|\Psi^p(t)\rangle. \tag{26}
$$

By using (25) and (26), we can obtain the expression of geometric phase as a function of the density matrix [27]. Moreover, we can prove that our expression is in agreement with nonunitary evolution [27, 34].

This is very important because the density matrix was introduced as a way of describing the quantum open system and the state of the open system is not completely known. In a general case, the state for the open system can always be written in many different ways as a probabilistic mixture of distinct but not necessarily orthogonal pure states. In contrast to the kinematic approach [33], an orthogonal pure initial state is used. Therefore, the approach to the mixed state may be in a special case.

5 Geometric Structure

From (21) and (24), we see that, when the evolving time $t_0 = 0$ and $t = 2\pi/\Omega_r$, the total phase is 2π. In this condition, the constant total phase may be dropped. Equation (23) becomes

$$
\begin{aligned}
\gamma_g &= -\mathrm{Im}\left(\int_{t_0}^{t} \frac{\langle\Psi(t)|d|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle}\right) \\
&= -\frac{1}{2}\int_{t_0}^{t}(1-\cos\alpha(t))d\beta(t),
\end{aligned}
\tag{27}
$$

which is called as Berry phase. It is interesting to formulate the Berry phase for the mixed state by using the language of differential geometry as done in the pure state case [35].

Under a local gauge transformation for the pure state, such as

$$|\psi(\eta)\rangle \to |\psi'(\eta)\rangle = e^{-i\alpha(\eta)}|\psi(\eta)\rangle, \tag{28}$$

we find that the nonunit vector state $|\Psi(\eta)\rangle$ in (22), describing the open system, can be expressed as

$$|\Psi(\eta)\rangle \to |\Psi'(\eta)\rangle = e^{-i\alpha(\eta)}|\Psi(\eta)\rangle. \tag{29}$$

Thus, one has

$$\begin{aligned}
\beta &= -\mathrm{Im}\frac{\langle\Psi(\eta)|\frac{\partial\Psi(\eta)}{\partial\eta_i}\rangle}{\|\Psi(\eta)\|^2}d\eta_i \\
\to \beta' &= -\mathrm{Im}\frac{\langle\Psi'(\eta)|\frac{\partial\Psi'(\eta)}{\partial\eta_i}\rangle}{\|\Psi'(\eta)\|^2}d\eta_i \\
&= -\mathrm{Im}\left\langle\frac{\Psi'(\eta)}{\|\Psi'(\eta)\|}\middle|\frac{\partial}{\partial\eta_i}\frac{\Psi'(\eta)}{\|\Psi'(\eta)\|}\right\rangle d\eta_i + \mathrm{Im}\left(\|\Psi'(\eta)\|\frac{\partial}{\partial\eta_i}\frac{1}{\|\Psi'(\eta)\|}d\eta_i\right) \\
&= -\mathrm{Im}\left\langle e^{-i\alpha(\eta)}\frac{\Psi(\eta)}{\|\Psi(\eta)\|}\middle| -i\frac{\partial\alpha(\eta)}{\partial\eta_i}e^{-i\alpha(\eta)}\frac{\Psi(\eta)}{\|\Psi(\eta)\|} + e^{-i\alpha(\eta)}\frac{\partial\Psi(\eta)}{\partial\eta_i}\frac{\Psi(\eta)}{\|\Psi(\eta)\|}\right\rangle d\eta_i \\
&= -\mathrm{Im}\left\langle\frac{\Psi(\eta)}{\|\Psi(\eta)\|}\middle|\frac{\partial}{\partial\eta_i}\frac{\Psi(\eta)}{\|\Psi(\eta)\|}\right\rangle d\eta_i + \frac{\partial\alpha(\eta)}{\partial\eta_i}d\eta_i \\
&= \beta + d\alpha,
\end{aligned} \tag{30}$$

where $\|\Psi(\eta)\| = \sqrt{\langle\Psi(\eta)|\Psi(\eta)\rangle}$ and β is called as one-form. In (30), we already use the fact that $\langle\Psi(\eta)|\Psi(\eta)\rangle$ is real. By using an exterior differentials, furthermore, the two-form can be written by

$$\begin{aligned}
\sigma &= d\beta = \frac{\partial\beta_j(\eta)}{\partial\eta_i}d\eta_i\wedge d\eta_j \\
&= -\mathrm{Im}\left(\frac{\partial}{\partial\eta_i}\frac{\Psi(\eta)}{\sqrt{\langle\Psi(\eta)|\Psi(\eta)\rangle}}, \frac{\partial}{\partial\eta_j}\frac{\Psi(\eta)}{\sqrt{\langle\Psi(\eta)|\Psi(\eta)\rangle}}\right)d\eta_i\wedge d\eta_j \\
&\quad -\mathrm{Im}\left(\frac{\Psi(\eta)}{\sqrt{\langle\Psi(\eta)|\Psi(\eta)\rangle}}, \frac{\partial^2}{\partial\eta_i\partial\eta_j}\frac{\Psi(\eta)}{\sqrt{\langle\Psi(\eta)|\Psi(\eta)\rangle}}\right)d\eta_i\wedge d\eta_j \\
&= -\mathrm{Im}\left(\frac{\partial}{\partial\eta_i}\frac{\Psi(\eta)}{\sqrt{\langle\Psi(\eta)|\Psi(\eta)\rangle}}, \frac{\partial}{\partial\eta_j}\frac{\Psi(\eta)}{\sqrt{\langle\Psi(\eta)|\Psi(\eta)\rangle}}\right)d\eta_i\wedge d\eta_j \\
&= \sigma_{ij}d\eta_i\wedge d\eta_j,
\end{aligned} \tag{31}$$

where

$$\sigma_{ij} = -\mathrm{Im}\left\langle\frac{\partial}{\partial\eta_i}\frac{\Psi(\eta)}{\|\Psi(\eta)\|}\middle|\frac{\partial}{\partial\eta_j}\frac{\Psi(\eta)}{\|\Psi(\eta)\|}\right\rangle. \tag{32}$$

The symplectic two-form σ is invariant under the local gauge transformations (28) and (29), such as

$$\sigma \to \sigma' = d\beta + d^2\alpha = d\beta = \sigma. \tag{33}$$

Therefore, the Berry phase is rewritten as

$$\gamma_g(\mathcal{C} = \partial S) = \int_S \sigma = \oint_{\mathcal{C}} \beta, \tag{34}$$

which is the area enclosed in the path traced out the surface of the Bloch sphere. Therefore, the Berry phase of mixed state is a geometric phase associated with a nonunit vector state for the open system carried around the closed curve $\mathcal{C}$.

6 Discussion and Conclusion

The Pantcharatnam phase and corresponding radius of Bloch sphere as the functions of both the fluctuating rate κ of magnetic field and the evolving time t are shown at Figs. 1 and

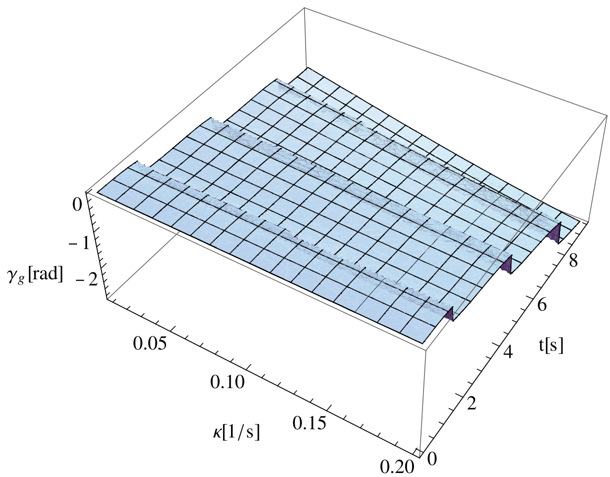


Fig. 1 Pantcharatnam phase in a fluctuating magnetic field as functions of the fluctuating rate κ and evolving time t with the parameters $\Omega = 1/\mathrm{s}$ and $\theta = \pi/4$, where the unit of time s is referred to multiplicative inverse of the atomic resonance frequency

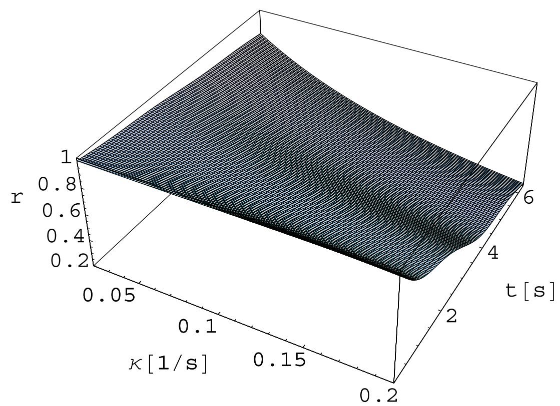


Fig. 2 Corresponding radius of Bloch sphere for Pantcharatnam phase as functions of the fluctuating rate κ and evolving time t with the same parameters as Fig. 1

Fig. 3 Two-dimensional parametric plots for radius of Bloch sphere as a function of the evolving time t for $\kappa = 0.05/\mathrm{s}$, $0.1/\mathrm{s}$, $0.15/\mathrm{s}$, $0.2/\mathrm{s}$ respectively with the same parameters as Figs. 1 and 2

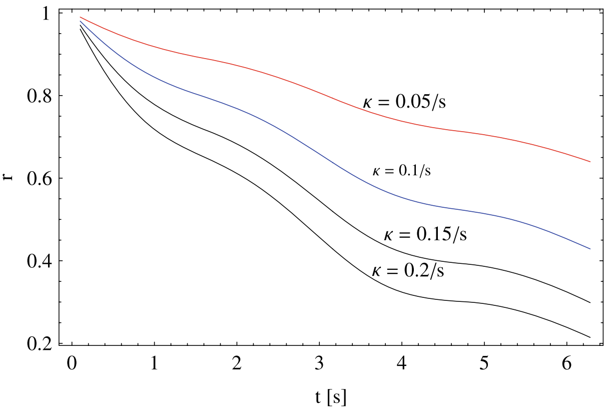
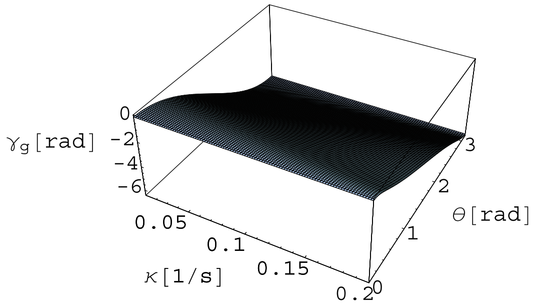


Fig. 4 Berry phase in a fluctuating magnetic field as functions of initial angle θ and the fluctuating rate κ with the parameter $\Omega = 1/\mathrm{s}$, where the phase is accumulated at the region $[0, t = 2\pi/\Omega_r]$ and the unit of time s is referred to multiplicative inverse of the atomic resonance frequency



2 respectively. Two-dimensional plot for the radius is shown at Fig. 3. We find that the phase oscillates accompanying the oscillation of the radius and decreases with creasing of κ and t, where the oscillation frequency Ω_r depends on the fluctuation of magnetic field and is different from the atomic resonance frequency Ω. These may be understood because, from (23) and (24), the phase depends strongly on the parameters of Bloch sphere, such as α, β and r, where the parameters are functions of $\cos(\Omega_r t)$, $\sin(\Omega_r t)$ and exponential decay (see (19)–(21)) and therefore the radius oscillates and decreases according to the evolving time (see Figs. 2 and 3). If $\kappa = 0$, $\Omega_r = \Omega$ and $r = 1$. Moreover, the oscillations disappear for both the Pantcharatnam phase and the Bloch parameters in this case. Therefore, the oscillations are resulted from the fluctuating magnetic field.

The Berry phase (see (27)) accumulated at the region $[0, t = 2\pi/\Omega_r]$ and Bloch radius at evolving time $t = 2\pi/\Omega_r$ are shown at Figs. 4 and 5 as functions of initial angle θ and fluctuating rate κ. Two-dimensional plots for the phase and the radius as functions of the fluctuating rate κ are shown at Figs. 6 and 7 respectively. We see that the radius at $t = 2\pi/\Omega_r$ oscillates according to θ and decreases with increasing of κ. However, it is interesting to note that the oscillations disappear for the Berry phase at the region $[0, t = 2\pi/\Omega_r]$ though it decreases with increasing of θ and κ, which is different from the radius at point of $t = 2\pi/\Omega_r$. It is known that, because the Berry phase is proportional to the area spanned in the Bloch parameter space (see (34)), it is insensitive to any oscillation (see Figs. 4 and 6). But the Berry phase is not simply given by the solid angle enclosed by the isolated spin resonances without the fluctuating field (see Figs. 6 and 7). It is worth noting that there doesn't exist the cyclic motion for the open physical system because there exists an exponential decay in the density matrix so that the field amplitudes describing the mixed state in the complex

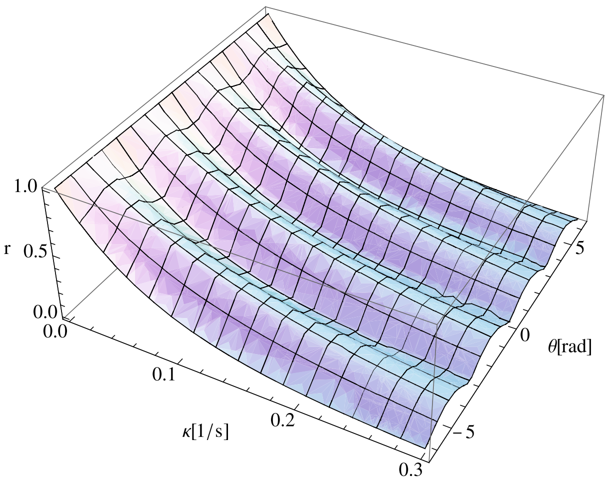


Fig. 5 Radius of Bloch sphere at evolving time $t = 2\pi/\Omega_r$ as functions of initial angle θ and the fluctuating rate κ with the same parameters as Fig. 4

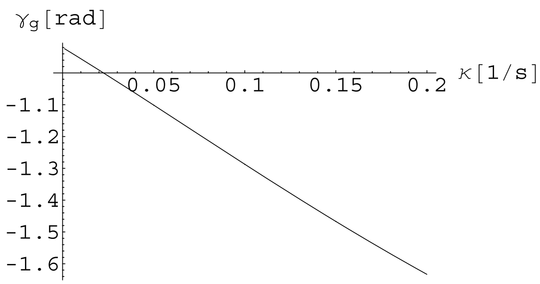


Fig. 6 Berry phase in a fluctuating magnetic field as a function of the fluctuating rate κ with the parameters $\Omega = 1/\mathrm{s}$ and $\theta = \pi/4$, where the phase is accumulated at the region $[0, t = 2\pi/\Omega_r]$ and the unit of time s is referred to multiplicative inverse of the atomic resonance frequency

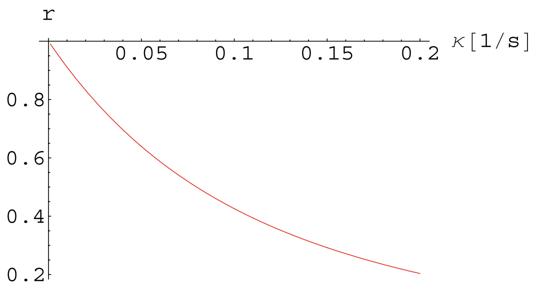


Fig. 7 Radius of Bloch sphere at evolving time $t = 2\pi/\Omega_r$ as a function of the fluctuating rate κ with the same parameters as Fig. 6

projective Hilbert space are a function of the exponential decay. If the fluctuating field disappears, the Bloch parameters become $r = 1$, $\alpha = \theta$ and $\beta = \Omega t$ according to (19)–(21). This implies the system is in the pure state. Furthermore, (27) will give the Berry Phase as $\gamma_g = -\pi(1 - \cos\theta)$ in this case. Therefore, our definition of geometric phase includes the result of the pure state.

Comparing the Berry phase with the Pantcharatnam phase for the open system, we find that, because the Pantcharatnam phase is a function of the evolving time, the phase is more strongly dependent on the fluctuating field than the Berry phase. In contrast to the Pantcharatnam phase, the Berry phase is accumulated at the region $[0, t = 2\pi/\Omega_r]$ of the evolving time and therefore it does not oscillate with the evolving time. However, it is worth noting that the dressed frequency of atomic resonance, $\Omega_r = \sqrt{\Omega^2 - \kappa^2}$, depends on the fluctuating rate. Therefore, the effect of fluctuating magnetic field on the Berry phase can not be neglected in our approach (see Figs. 6 and 7). This is different from that of [30], where the contributions from this effect are proportional to κ^2 so that this effect can be neglected at the level of approximation.

In conclusion, a method is suggested to calculate the geometric phase in the fluctuating magnetic field. We find that the Berry phase is not the same for the pure state as for the mixed state. Furthermore, a simple geometric interpretation is analyzed for the dressed spin resonances in terms of the Bloch parameter space.

In comparison with the previous treatment to the fluctuating magnetic field in the calculation of geometric phase [30], our results include obviously the effect of the fluctuating magnetic field. In contrast to the other previous definitions for the geometric phase in open system [21–26, 33], our approach is to emphasize its geometric structure, where the simple structure is exploited so that any state can be described as (generally un-normalized) vector in a Bloch sphere. Thus, the geometric phase is only dependent on the smooth (open or closed) curve in the complex projective Hilbert space. By renormalizing this vector, we define the geometric phase as the area enclosed in the path traced out on the surface of the sphere.

From Figs. 1–7, we see that our geometric phase is a continuous function of the evolving time, which is in contrast to the quantum jump and kinematic approaches where there existed some discontinuous points [36] in the geometric phase. It may be difficult to understand the discontinuities in physics because all elements of the density matrix, describing the quantum open physical system, are continuous functions.

References

1. Pancharatnam, S.: Proc. Indien Acad. Sci. A **44**, 1225 (1956)
2. Berry, M.V.: Proc. R. Soc. A **392**, 45 (1984)
3. Aharonov, Y., Anandan, J.: Phys. Rev. Lett. **58**, 1593 (1987)
4. Wang, Z.S., Kwek, L.C., Lai, C.H., Oh, C.H.: Phys. Lett. A **359**, 608 (2006)
5. Wang, Z.S., Kwek, L.C., Lai, C.H., Oh, C.H.: Eur. Phys. J. D **33**, 285 (2005)
6. Wang, Z.S., Kwek, L.C., Lai, C.H., Oh, C.H.: Phys. Scr. **75**, 494 (2007)
7. Wang, Z.S., Kwek, L.C., Lai, C.H., Oh, C.H.: Phys. Lett. A **372**, 775 (2008)
8. Khanna, G., Simon, R., Mukunda, N.: Ann. Phys. **253**, 55 (1997)
9. Byrd, M.: J. Math. Phys. **39**, 6125 (1998)
10. Strahov, E.: J. Math. Phys. **42**, 2008 (2001)
11. Suter, D., Mueller, K.T., Pine, A.: Phys. Rev. Lett. **60**, 1218 (1988)
12. Bhandari, R., Samuel, J.: Phys. Rev. Lett. **60**, 1211 (1988)
13. Chiao, R.Y., Antaramian, A., Ganga, K.M., Jiao, H., Wilkinson, S.R., Nathel, H.: Phys. Rev. Lett. **60**, 1214 (1988)
14. Zanardi, P., Rasetti, M.: Phys. Lett. A **264**, 94 (1999)
15. Ekert, A., Ericsson, M., Hayden, P., Inamori, H., Jones, J.A., Oi, D.K.L., Vedral, V.: J. Mod. Opt. **47**, 2501 (2000)
16. Wang, Z.S., Wu, C., Feng, X.-L., Kwek, L.C., Lai, C.H., Oh, C.H.: Phys. Rev. A **76**, 044303 (2007)
17. Uhlmann, A.: Rep. Math. Phys. **24**, 229 (1986)
18. Samuel, J., Bhandari, R.: Phys. Rev. Lett. **60**, 2339 (1988)
19. Ellinas, D., Dupertuis, S.M., Dupertuis, M.A.: Phys. Rev. A **39**, 3228 (1989)
20. Dattoli, G., Mignani, R., Torre, A.: J. Phys. A **23**, 5795 (1990)
21. Sjöqvist, E., Pati, A.K., Ekert, A., Anandan, J.S., Ericsson, M., Oi, O.K.L., Vedreal, V.: Phys. Rev. Lett. **85**, 2845 (2000)
22. Carollo, A., Fuentes-Guridi, I., Franca Santos, M., Vedral, V.: Phys. Rev. Lett. **90**, 160402 (2003)
23. Fonseca Romero, K.M., Aguiar, A.C., Thomaz, M.T.: Physica A **307**, 142 (2002)
24. Nazir, A., Spiller, T.P., Munro, W.J.: Phys. Rev. A **65**, 042303 (2003)
25. Whitney, R.S., Gefen, Y.: Phys. Rev. Lett. **90**, 190402 (2003)
26. Whitney, R.S., et al.: Phys. Rev. Lett. **94**, 070407 (2005)

27. Wang, Z.S., Kwek, L.C., Lai, C.H., Oh, C.H.: Europhys. Lett. **74**, 958 (2006)
28. Carollo, A., Massimo Palma, G., Lozinski, A., Santos, M.F., Vedral, V.: Phys. Rev. Lett. **96**, 150403 (2006)
29. Wang, Z.S., Wu, C., Feng, X.-L., Kwek, L.C., Lai, C.H., Oh, C.H.: Phys. Rev. A **75**, 024102 (2006)
30. De Chiara, G., Palma, M.: Phys. Rev. Lett. **91**, 090404 (2003)
31. Loreti, F.N., Balantekin, A.B.: Phys. Rev. D **50**, 4762 (1994)
32. Lindblad, G.: Commun. Math. Phys. **48**, 119 (1976)
33. Tong, D.M., Sjöqvist, E., Kwek, L.C., Oh, C.H.: Phys. Rev. Lett. **93**, 080405 (2004)
34. Mukunda, N., Simon, R.: Ann. Phys. **228**, 205, 269 (1993)
35. Simon, B.: Phys. Rev. Lett. **51**, 2167 (1983)
36. Yi, X.X., Tong, D.M., Wang, L.C., Kwek, L.C., Oh, C.H.: Phys. Rev. A **73**, 052103 (2006)